

Resonance effects on the crossover of bosonic to fermionic superfluidity

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Feshbach scattering resonances are being utilized in atomic gases to explore the entire crossover region from a Bose-Einstein Condensation (BEC) of composite bosons to a Bardeen-Cooper-Schrieffer (BCS) of Cooper pairs. Several theoretical descriptions of the crossover have been developed based on an assumption that the fermionic interactions are dependent only on the value of a single microscopic parameter, the scattering length for the interaction of fermion particles. Such a picture is not universal, however, and is only applicable to describe a system with an energetically broad Feshbach resonance. In the more general case in which narrow Feshbach resonances are included in the discussion, one must consider how the energy dependence of the scattering phase shift affects the physical properties of the system. We develop a theoretical framework which allows for a tuning of the scattering phase shift and its energy dependence, whose parameters can be fixed from realistic scattering solutions of the atomic physics. We show that BCS-like nonlocal solutions may build up in conditions of resonance scattering, depending on the effective range of the interactions.

PACS numbers: 03.75.Ss

Introduction

Fermion pairing is a fundamental concept in the manifestation of non-trivial ground states in condensed-matter physics. Condensation of composite fermions, viewed as bound states of an electron and an even number of vortices, is a possible explanation of Integer and Fractional Quantum Hall Effects in the highly degenerate two-dimensional electron gas, the vortices corresponding to the fractionally charged quasi-particles of Laughlin’s theory [1]. Exciton formation and possibly Bose-Einstein condensation [2] in semiconductor structures is one more example [3]. Cooper pairing resulting from correlations in momentum space is the mechanism determining the BCS-type superconductivity in metallic compounds [4], while strong correlations [5] and real-space pairing with a short coherence length characterize the high-temperature superconductors, where pair correlations manifest in the opening of a pseudogap well before the superconducting transition [6].

The achievement of quantum degeneracy in atomic Fermi gases of ^{40}K and ^6Li after cooling in dipolar optical traps [7, 8, 9, 10, 11, 12], and of accurate control of the interactions by means of Feshbach resonances [13], has made these concepts accessible in atomic-physics experiments. This has initiated an exploration of an intriguing system which should reveal the predicted resonance superfluidity [14, 15, 16] and the nature of BCS pairing

with large attractive interaction [17].

In these experiments, the magnetic field is changed to tune the position of a bound state in a closed channel in the interatomic potential with respect to the threshold of zero scattering energy. The zero energy is defined by the asymptotic value of an open channel potential in which the colliding atoms enter. At exact resonance, the bound state connects with the zero energy scattering solution and the scattering length is infinity. Variations of the magnetic field B with respect to the resonance value B_0 can be converted into an effective detuning $\nu = (B - B_0)\Delta\mu^{\text{mag}}$, with $\Delta\mu^{\text{mag}}$ the relative magnetic moment between the closed-channel bound state and the open-channel threshold. As displayed in Fig. 1 for the case of ^6Li , positive (negative) detunings with respect to the resonance correspond to effectively attractive (repulsive) interactions resulting in negative (positive) scattering lengths a . The Feshbach mechanism evidently involves a separation of energy and length scales between the background and the resonant behavior, the former driven by the value a_{bg} at very large ν , and the latter dictated by the position and width $\Delta\nu = \Delta B \Delta\mu^{\text{mag}}$ of the resonance. We point out that certain physical systems have large values for the background scattering due to the presence of a potential or shape resonance in the open channel, and a multiple resonance model should be developed in that case [18].

The side of the resonance corresponding to repulsive

interactions ($a > 0$) gives rise to a rich and complex quantum system. Here, the formation of weakly bound molecular states of two fermions has been obtained [19, 21, 22] and their Bose-Einstein condensation (BEC) [23, 24, 25, 26] has been observed. On the other side of the Feshbach resonance ($a < 0$) where the interactions are attractive, experimental measurements are being made on the superfluid paired state that is likely associated with a strong-coupling version of the Bardeen-Cooper-Schrieffer (BCS) theory.

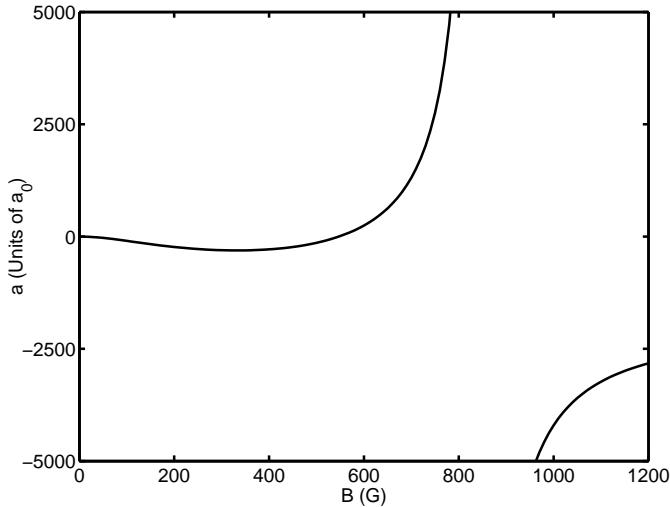


FIG. 1: The broad Feshbach resonance in ${}^6\text{Li}$ [16]. The scattering length a is shown as a function of magnetic field, for the $(f, m_f) = (1/2, -1/2)$ and $(1/2, 1/2)$ mixed spin channel.

By continuous tuning of the interactions across the Feshbach resonance, the whole crossover region from the relatively simple limits of a BEC of preformed pairs and a BCS-like superfluid state is accessible in the experiment. There are many open questions in the understanding of the region in between, including the relevant correlations in the system and the nature of the order parameter [27]. Several theoretical descriptions of the BCS-BEC crossover in Fermi gases with Feshbach resonances [28, 29, 30, 31] have been developed, inspired by the early seminal works by Leggett [32] and Nozières and Schmitt-Rink [33], and subsequent developments in the context of high- T_c superconductivity.

The crossover theories can be divided in two general classes: studies in which the interactions are parametrized by the scattering length [34], and those that explicitly include the presence of molecular states of two fermions and the hybridization of these paired states with their unpaired counterpart [35]. Both approaches share the idea that the formation of Cooper pairs and their condensation to the coherent superfluid state do not occur at the same time. There are formal connections based on either application of the Hubbard-Stratonovich transformation to the functional integral [36], or to the dressing of the open and closed channel solutions [27] which demonstrate the equivalence of the approaches when the

resonance state has a sufficiently short lifetime.

Two relevant questions arise in the quantitative description of the crossover behavior in atomic gases. First, an adequate many-body theory providing the crossover order parameter must be able to recover the results of the four-fermion scattering problem entering the Feshbach physics [37]. Second, the energy dependence of the scattering phase shift may be important over the energy range of the Fermi energy [38], and thus the width of the resonance becomes a relevant parameter of the theory in addition to a . In the unitarity limit $|a| \rightarrow \infty$, where the scattering length is meaningless, the universal behavior of the thermodynamic properties emerges only in the case of broad resonances [39], and otherwise the specific details of the resonance will enter. All these aspects pose stringent conditions to the choice of suited theoretical approaches. Along these lines, numerical methods would be a very useful guideline to assess the validity of approximate schemes and their underlying physics.

In this work we focus on the issue of the energy dependence of the scattering phase shift in the unitarity limit, anticipating that the general case of a BCS-BEC crossover driven by two independent parameters (the detuning and width of the resonance) will lead to an intriguing and nontrivial phase diagram. Along these lines, we develop a model that is able to interpolate between the two limits of a broad resonance, where the scattering length resulting from low-energy resonance renormalization completely encapsulates the interaction behavior, and of a “high-quality” resonance in a broad Fermi sea, where large values of scattering energies need to be sampled. We model an interaction potential that is composed of a short-range attractive well followed by a barrier structure, where scattering length and resonance width can be independently tuned. We use such a model potential to show that a nonlocal BCS-like superfluid ground state may emerge under non trivial conditions. The results are indicative at this stage, as they are obtained within a mean-field approach that is not necessarily valid in the regime of strong interactions. However, it is a necessary precursor to perform such calculations to guide the more complicated Quantum Monte Carlo (QMC) simulations, that are under way.

The structure of the paper is as follows. We first develop an explicit microscopic model based on a potential-well and a potential-barrier to encapsulate the essential physics of the Feshbach resonance. We then derive the nonlocal BCS equations for this system, assuming a closure of the hierarchy of many-body correlations at the BCS level. Finally, the BCS equations are self-consistently solved and the results discussed in view of their application to the development of a more detailed QMC simulation needed to capture the effects of many-particle correlations.

A well-barrier model of the Feshbach resonance

We model the presence of a Feshbach resonance by the interaction potential of the form displayed in Fig. 2,

$$V(r) = \begin{cases} -V_0 & r < r_0 \\ V_1 & r_0 < r < r_1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

that is characterized by an attractive well with depth V_0 and width r_0 and a barrier with height V_1 and width $r_1 - r_0 \equiv r_w$. This well-barrier model allows us to incorporate the essential energy-dependence of the scattering physics into a single-channel scattering scheme.

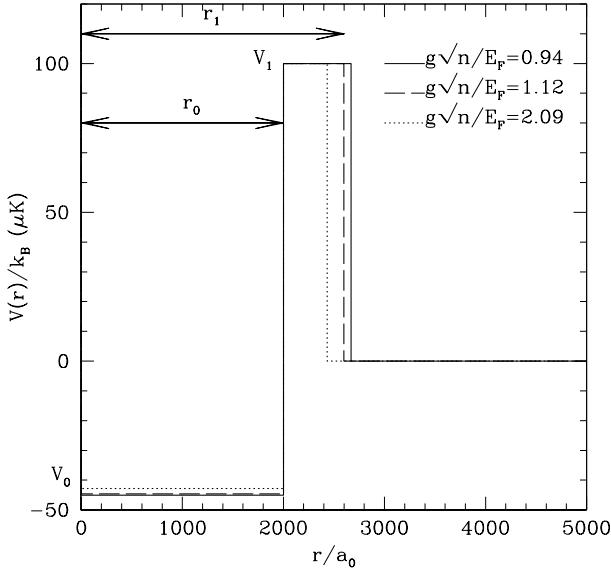


FIG. 2: A well-barrier model that allows for the independent tuning of the scattering length a and of the resonance width g . The well can support resonant states with a width dependent on the tunneling through the potential barrier. The situations shown correspond to the values used in subsequent simulations which all have $a = 5000a_0$, where a_0 is the Bohr radius. Different cases are represented by the solid, dashed and dotted lines correspond to increasing values of the resonance width g (cases 1, 4, and 8 in Table I).

The condition for diluteness of the gas is that the range of the interatomic potential should be much less than the interparticle spacing, a condition that we assume to be fulfilled through this paper. Explicitly this requires us to satisfy at all times $nr_0^3 \ll 1$, where n is the particle density. It is important to emphasize that this is distinct from the condition $na^3 \ll 1$ associated with the convergence of the perturbation theory of the gas in the contact scattering approximation. In the vicinity of the resonance, we may have the situation in which $|a| > r_0$ so that we simultaneously have a system which is dilute in the sense that $nr_0^3 \ll 1$ but unitarity limited in the sense that $na^3 > 1$. We can furthermore distinguish the two

different regimes of narrow and broad resonance based on a comparison of the Fermi energy $E_F = \hbar^2 k_F^2 / 2m$ with $k_F = (3\pi^2 n)^{1/3}$ with the resonance width $\Delta\nu$. The resonance width $\Delta\nu$ can be expressed in terms of the matrix element g for the coupling between the closed and open channels as $\Delta\nu = g\sqrt{n}$, that enters the resonance-superfluidity Hamiltonian, as pointed out in Ref. [40].

The parameters of the model potential can in principle be adjusted to reproduce the scattering properties of an atomic sample, as *e.g.* they are determined from collision experiments or from full coupled-channel calculations [16]. We instead perform here a study over a set of parameters chosen in order to illustrate the important physical behavior of the system. We fix the scattering length to a large and positive value, $a = 5000 a_0$, with a_0 the Bohr radius. We ensure this is large compared to the interparticle spacing as determined from the density which we take as $n = 1.054 \times 10^{14} \text{ cm}^{-3}$. The large positive scattering length corresponds to the region of unitarity limited behavior considered just on the BEC-side of the resonance. With these fixed constraints we vary the resonance width crossing the full region of broad to narrow values. To this aim, we first set the range r_0 of the potential to a value that is small enough to satisfy the condition $nr_0^3 \ll 1$. This is the case for $r_0 = 2000 a_0$ giving $nr_0^3 = 0.125$. We intentionally choose this value here so that it is not too small which will allow us later to see on a single energy scale both the scattering and many-body effects. Then, we solve for V_1 (the barrier height) and r_1 (the barrier width) which are not independent and can be tuned in order to change the tunneling rate through the barrier and therefore $\Delta\nu$.

This requires solving the scattering problem for $V(r)$. The two-body scattering function $\Psi(r)$ is given in the three regions by

$$\Psi_w(r) = \begin{cases} \Psi_w(r) & r < r_0 \\ \Psi_b(r) & r_0 < r < r_1 \\ \Psi_f(r) & \text{otherwise} \end{cases}, \quad (2)$$

with

$$\begin{aligned} r\Psi_w(r) &= \sin(k_w r) \\ r\Psi_b(r) &= A_1 \exp[-k_b(r - r_0)] + B_1 \exp[k_b(r - r_0)] \\ r\Psi_f(r) &= A_2 \sin(kr) + B_2 \cos(kr) \end{aligned} \quad (3)$$

and

$$\begin{aligned} k_w &= \sqrt{k^2 + mV_0/\hbar^2}, \\ k_b &= \sqrt{mV_1/\hbar^2 - k^2}. \end{aligned} \quad (4)$$

The values of $A_{1,2}$ and $B_{1,2}$ are determined after imposing the usual boundary conditions $\Psi_w(r_0) = \Psi_b(r_0)$, $\Psi'_w(r_0) = \Psi'_b(r_0)$, $\Psi_b(r_1) = \Psi_f(r_1)$, and $\Psi'_b(r_1) = \Psi'_f(r_1)$, *i.e.* ensuring the continuity of the wave function and of its derivative at the boundaries r_0 and r_1 . The two-body T -matrix can then be extracted from the

scattering solution

$$T(k) = \frac{4\pi\hbar^2 B_2}{mk(iB_2 - A_2)}. \quad (5)$$

We can now proceed to link the parameters of the potential with the parameters of the many-body theory we wish to solve. The first parameter, the scattering length a is determined simply from its definition applied to Eq. (5),

$$\frac{4\pi\hbar^2 a}{m} = \lim_{k \rightarrow 0} T(k). \quad (6)$$

The width of the resonance is related physically to a matrix element between a continuum scattering state and a high-quality resonant state in the inner well. In practice, such a matrix element is determined by the tunneling rate through the barrier. In order to make the link explicit we present the usual relation between the T and S -matrices in scattering theory

$$T(k) = \frac{2\pi\hbar^2 i}{mk} [S(k) - 1] \quad (7)$$

and give the Feshbach form for the S -matrix in the one resonance parametrization

$$S(k) = e^{-2ika_{\text{bg}}} \left[1 - \frac{2ik|g|^2}{-\frac{4\pi\hbar^2}{m}(\nu - \frac{\hbar^2 k^2}{m}) + ik|g|^2} \right]. \quad (8)$$

Solving Eqs. (7) and (8) for g gives

$$|g|^2 = -\frac{8\pi\hbar^4}{m^2 R_{\text{eff}}} \quad (9)$$

where we have removed the background $a_{\text{bg}} = 0$ and the effective range is by definition $R_{\text{eff}} \equiv -(4\pi\hbar^2/m)(d^2T(k)^{-1}/dk^2)|_{k=0}$ which is determined explicitly from the scattering model Eq. (5) [41]. Note this form in Eq. (9) agrees with the expression in [38]. The dimensionless width in units of the Fermi energy is then the ratio $g\sqrt{n}/E_F$ as previously discussed.

We list in the second column of Table I the set of $g\sqrt{n}/E_F$ values that, together with the fixed condition $a = +5000 a_0$, define from here on our case studies. The corresponding values of r_1 and V_0 are reported in the third and fourth column. We illustrate in Fig. 3 the wavefunctions $r\Psi(r)$ that are found for the three $V(r)$ potentials reported in Fig. 2 and labeled as cases 1, 8 and 10 in Table I, and for the incoming kinetic energy $E_k/k_B = 1 nK$. While it is expected that the wavefunction should vanish near $r = +5000 a_0$, corresponding to the value of a , it is clear that a knee develops in the probability amplitude for the scattered wave in the spatial region within the barrier. This feature becomes more pronounced as the resonance width decreases.

TABLE I: Parameters for the model potential $V(r)$ in Fig. 2. The values of r_1 and V_0 are reported in the third and fourth column, that yield the different values of $g\sqrt{n}/E_F$ in the second column with $a = 5000a_0$ fixed. The various cases are numbered as in the first column. The remaining model parameters are $r_0 = 2000a_0$ and $V_1 = 100\mu K$.

| Case | $g\sqrt{n}/E_F$ | r_1/r_0 | $V_0(\mu K)$ |
|------|-----------------|-----------|--------------|
| 1 | 0.94 | 1.3345 | 45.0581 |
| 2 | 0.99 | 1.3236 | 44.9236 |
| 3 | 1.05 | 1.312 | 44.7673 |
| 4 | 1.12 | 1.2994 | 44.5806 |
| 5 | 1.32 | 1.2712 | 44.0898 |
| 6 | 1.48 | 1.255 | 43.7547 |
| 7 | 1.71 | 1.237 | 43.3292 |
| 8 | 2.09 | 1.217 | 42.7804 |
| 9 | 2.96 | 1.1935 | 42.0159 |

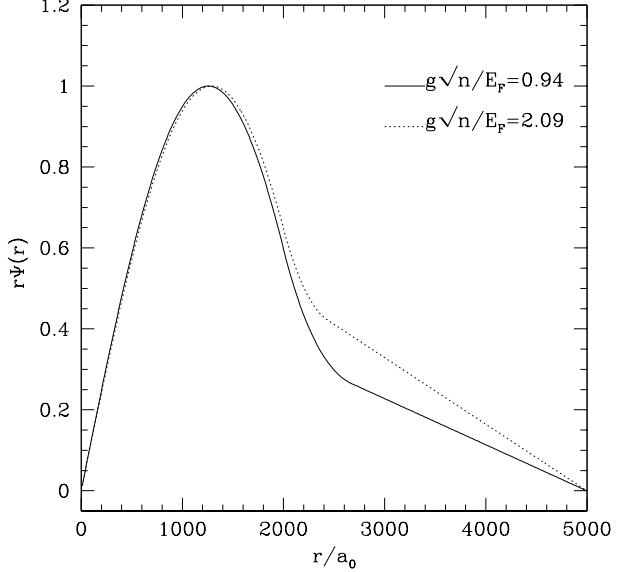


FIG. 3: The two-body wave function $r\Psi(r)$ corresponding to the case potentials $V(r)$ 1 and 8 in Fig. 2 and in Tab. I: $g\sqrt{n}/E_F = 0.94$ (solid line), 2.09 (dotted line). The incoming kinetic energy is $E_k/k_B = 1 nK$.

The nonlocal BCS equations

We can now utilize our analysis of the two-body physics for this model potential as an input to the many-body theory. In particular we wish to find the ground-state of the Fermi gas interacting via $V(r)$. As mentioned in the introduction, we aim in this paper to solve the variational BCS scheme for the case of a nonlocal potential interaction. Since our nonlocal potential allows us to explore the consequences of scattering resonances, this is a suitable foundation as eventual input to QMC schemes.

Such schemes are necessary to include the many-particle correlations which go beyond the pairing fields studied here at the mean-field level in order to determine the unitarity limit properties of the gas.

The ground state is determined within the variational BCS scheme by means of the wave-function [4]:

$$|\Phi_0\rangle = \Pi_{\mathbf{k}}(u_{\mathbf{k}} + v_{\mathbf{k}} a_{\mathbf{k},\uparrow}^+ a_{-\mathbf{k},\downarrow}^+) |0\rangle, \quad (10)$$

where $a_{\mathbf{k},\sigma}^+$ is the creation operator for electrons of spin σ . The normalization of Φ_0 leads to the condition $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$. The expected value of the ground-state energy reads:

$$\begin{aligned} E_0 &= \sum_{\mathbf{k}} 2\epsilon_{\mathbf{k}}|v_{\mathbf{k}}|^2 + \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} u_{\mathbf{k}} v_{\mathbf{k}}^* u_{\mathbf{k}'} v_{\mathbf{k}'}^* \\ &+ \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} u_{\mathbf{k}} u_{\mathbf{k}'}^* v_{\mathbf{k}'} v_{\mathbf{k}'}^* \end{aligned} \quad (11)$$

where $\epsilon_{\mathbf{k}} = \hbar^2 \mathbf{k}^2 / 2m$.

The summations in Eq. (11) can be converted into one-dimensional integrals after performing the angular integrations. For the generic function $F(\mathbf{k}')$, in our situation of isotropic pairing this amounts to a substitution

$$\sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} F(\mathbf{k}') \rightarrow \frac{1}{(2\pi)^3} \int dq q^2 V(k, q) F(q), \quad (12)$$

with $V(k, q)$ determined from the three-dimensional Fourier transform of the spatial potential

$$\begin{aligned} V(k, q) &= \frac{2\pi}{kq} (V_0 + V_1) \left[\frac{\sin r_0 |k+q|}{|k+q|} - \frac{\sin r_0 |k-q|}{|k-q|} \right] \\ &- V_1 \left[\frac{\sin r_1 |k+q|}{|k+q|} - \frac{\sin r_1 |k-q|}{|k-q|} \right]. \end{aligned} \quad (13)$$

The BCS solution is obtained by minimizing the free energy $f = \langle \Phi_0 | \hat{H} | \Phi_0 \rangle - \mu \langle \Phi_0 | \hat{N} | \Phi_0 \rangle$ with respect to the variational parameters u_k, v_k . The chemical potential μ is determined by the constraint to have the correct particle density. The two resulting equations to be self-consistently solved correspond respectively to the isotropic superfluid gap and the particle density and are given by

$$\Delta(k) = \frac{1}{(2\pi)^3} \int dq q^2 V(k, q) \frac{\Delta(q)}{2E(q)} \quad (14)$$

$$n = \frac{1}{(2\pi)^3} \int d\mathbf{k} \left(1 - \frac{\xi(\mathbf{k})}{E(\mathbf{k})} \right) \quad (15)$$

In Eqs. (14)-(15), the excitation energy $E(k) = \sqrt{\Delta(k)^2 + \xi(k)^2}$ is expressed in terms of the gap function and of the single-particle energy $\xi(\mathbf{k})$

$$\xi(\mathbf{k}) = \epsilon_{\mathbf{k}} - \mu + \frac{1}{2} V_{k=q=0} n - \sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} \left(1 - \frac{\xi(\mathbf{k}')}{E(\mathbf{k}')} \right), \quad (16)$$

The second and third terms in Eq. (16) are the Hartree-Fock corrections to the single-particle self-energy, that manifest as a correction to the chemical potential.

We iteratively solve Eqs. (14) and (15) until self-consistency is obtained. Since two different length scales, r_0 and k_F^{-1} , are involved, integrations on Eqs. (14) and (15) are executed by means of Gaussian integration in three different grids of k -points that are in the ranges of k_F, r_0^{-1} , and $+\infty$ (the latter is performed after changing to the inverse variable $1/k$). We have typically used a total of $N_k = 1200$ grid points for this purpose.

The Bogoliubov u_k and v_k functions resulting from the energy minimization are finally evaluated through the expressions

$$u_k = \text{sgn}(\Delta(k)) \sqrt{\frac{1}{2} \left(1 + \frac{\xi(k)}{E(k)} \right)} \quad (17)$$

$$v_k = \sqrt{\frac{1}{2} \left(1 - \frac{\xi(k)}{E(k)} \right)}. \quad (18)$$

Self-consistent Results

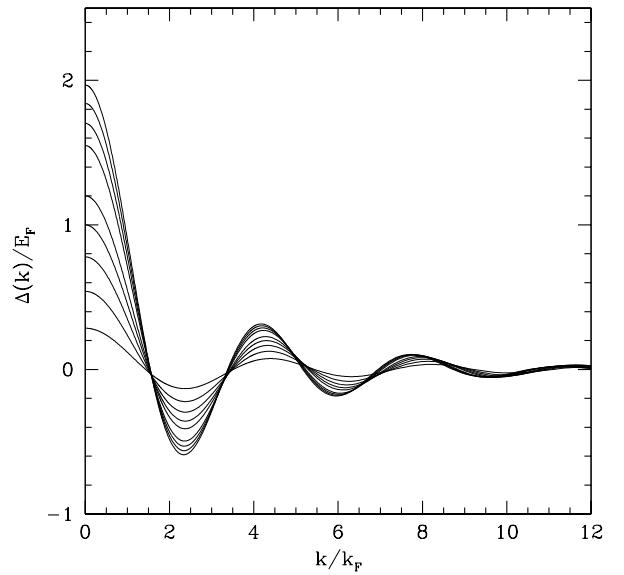


FIG. 4: $\Delta(k)$ in the unitarity limit with $a = 5000 a_0$, as a function of the resonance width $g\sqrt{n}/E_F$. Curves with increasing values of $\Delta(k=0)$ correspond to decreasing values of $g\sqrt{n}/E_F$, from case 9 to case 1 in Table I.

The gap function $\Delta(k)$ is displayed in Fig. 4 for different values of $g\sqrt{n}/E_F$. The damped oscillatory behavior on the scale of $1/r_0$ is a manifestation of the pairing potential, with a wavelength and damping coefficient that are almost independent of the resonance width.

The behavior of $\Delta_0 \equiv \Delta(k=0)$ as a function of $g\sqrt{n}/E_F$ is summarized by the squares in Fig. 5, together

with the behavior of the chemical potential (circles) following the gap. Fig. 5 demonstrates that a BCS-like solution emerges while the resonance shrinks on the scale of E_F , even on the BEC-side of the resonance, where the BCS-variational ansatz is not expected to give a complete description. These results suggest that a high-quality resonance leads to a nonuniversal regime in the unitarity limit. We have checked the consistency of this picture by computing the self-consistent solutions to Eqs. (14)-(15) in the case of narrower wells. The narrowing of the

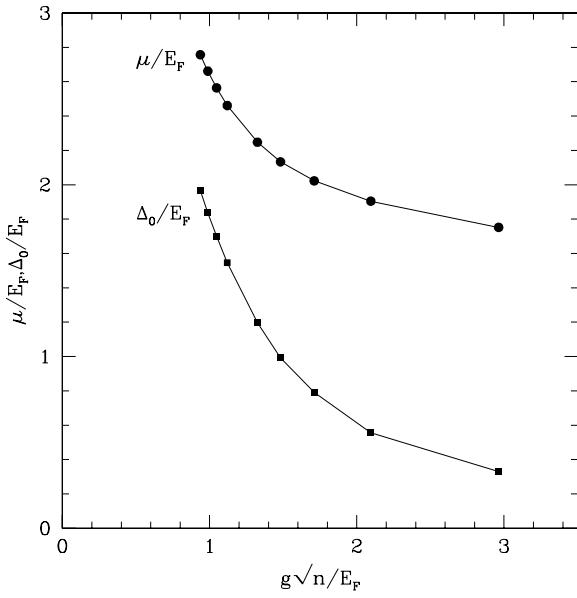


FIG. 5: $\Delta_0 \equiv \Delta(k = 0)$ (squares) and μ (circles) as functions of $g\sqrt{n}/E_F$, with $a = +5000 a_0$ fixed.

resonance seems to increase the level of the interactions that are responsible for the superfluid pairing. This is quantified in the ground-state energy (11), whose values are seen to become larger and negative with decreasing $g\sqrt{n}/E_F$ (see Tab. II).

TABLE II: Values of the total energy E_0 (first column), chemical potential μ (second column) and of the gap $\Delta_0 \equiv \Delta(k = 0)$ at $k = 0$ for the different values of $g\sqrt{n}/E_F$ in Tab. I.

| $g\sqrt{n}/E_F$ | E_0/E_F | μ/E_F | Δ_0/k_F |
|-----------------|-----------|-----------|----------------|
| 0.94 | -1.286760 | 2.757061 | 1.967685 |
| 0.99 | -1.240605 | 2.661722 | 1.838010 |
| 1.05 | -1.191324 | 2.563657 | 1.699060 |
| 1.12 | -1.138285 | 2.461363 | 1.545724 |
| 1.32 | -1.021072 | 2.247637 | 1.195665 |
| 1.48 | -0.953917 | 2.133496 | 0.993021 |
| 1.71 | -0.877986 | 2.012829 | 0.771605 |
| 2.09 | -0.816665 | 1.910900 | 0.556854 |
| 2.96 | -0.687847 | 1.751602 | 0.330776 |

Further insight can be obtained from the analysis of the momentum distribution $n(k)$

$$n(k) = |v_k|^2, \quad (19)$$

that is reported in Fig. 6. In agreement with the conclusions from Fig. 5, larger values of $g\sqrt{n}/E_F$ correspond larger values of the jump at k_F and thus to a normal Fermi-gas character. The momentum distribution typical of a BCS superfluid develops with decreasing values of $g\sqrt{n}/E_F$.

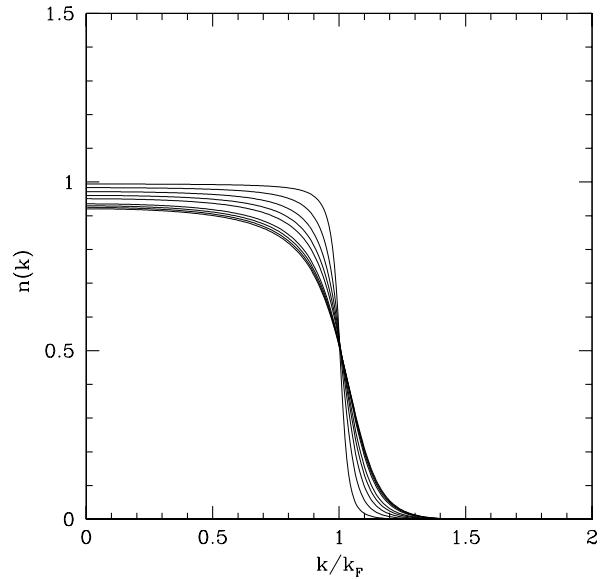


FIG. 6: Momentum distribution of the Fermi gas in the unitarity limit with $a = 5000 a_0$, as a function of the resonance width $g\sqrt{n}/E_F$. Curves with increasing values of $n(k = 0)$ correspond to increasing values of $g\sqrt{n}/E_F$, from case 1 to case 9 in Tab. I.

The superfluid state is also signaled by the emergence of a peak in the pair distribution function, that is

$$g(r) = g_{HF}(r) + g_p(r), \quad (20)$$

where the Hartree-Fock contribution is

$$g_{HF}(r) = \frac{1}{4} - \left(\frac{1}{(2\pi)^3 n} \right)^2 \int d\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{r}} |v_{\mathbf{k}}|^2 \quad (21)$$

and the pairing term

$$g_p(r) = + \left(\frac{1}{8\pi^3 n} \right)^2 \int d\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{r}} u_{\mathbf{k}} v_{\mathbf{k}}. \quad (22)$$

The peak emerges on the scale of the interparticle distance $r_s a_0$ with $r_s = (4\pi n a_0^3 / 3)^{-1/3}$, and becomes better defined as long as the quality of the resonant mode increases.

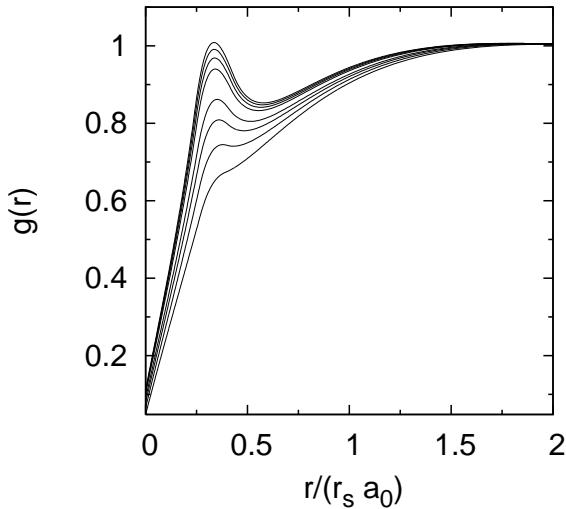


FIG. 7: Pair distribution function of the Fermi gas vs. the interparticle distance r_{sa0} in the unitarity limit with $a = 5000 a_0$. Curves with increasing values of peak height correspond to decreasing values of $g\sqrt{n}/E_F$, from case 2 to case 8 in Tab. I.

Conclusions

We have determined a solution to the many-body theory for the case of a Feshbach resonance within the BCS-variational ansatz. The situation we considered had a positive large scattering length corresponding to a near

unitarity limited system on the BEC side of the resonance. The dependence of the scattering phase shift on energy was shown to be important in the self-consistent solutions for the mean-fields. The solutions were found to depend sensitively on the resonance width. We could vary all of these parameters within our model by modifying the properties of an explicit and simple potential model consisting of a potential well and potential barrier. Due to the fact that the solution we obtained was for a nonlocal system, the resulting many-body theory did not suffer from a formal ultraviolet divergence and was therefore automatically renormalized. The self-consistent pairing field had to be determined at each wavevector value, illuminating the nonlocal character of the solution. The fact that we find a BCS solution for a positive value of the scattering length can be understood from the total self-energy of the system, which can be negative for narrow Feshbach resonances.

We emphasize that around the resonance, the BCS-variational ansatz is not expected to give a complete description, although this is the exclusively applied framework in which the BCS-BEC crossover theories have been implemented to date. It is a necessary precursor to explore and characterize the solutions we have found at the mean-field level as inputs to a Quantum Monte Carlo numerical method, which is able to encapsulate the many-particle correlations which we have dropped. This is an important problem where it is anticipated that the unitarity limit of the quantum system will emerge when the scattering resonance is broad, but that a high-quality resonance will lead to a nonuniversal regime which depends on microscopic resonance parameters.

M. H. would like to acknowledge discussions with S. Giorgini and support from the National Science Foundation. S. D. P. and M. L. C. would like to thank G. La Rocca for useful discussions. M. L. C. acknowledges support from the CRS-BEC of Trento, where has benefited from discussions with S. Giorgini and S. Stringari, and from INFM through the project ATOMOPTICS. S. K. acknowledges support from the Netherlands Organisation for Scientific Research (NWO).

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